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The analytical calculations of nonlinear dynamics of Kerr medium

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Abstract

This paper aims to study some calculations related to the nonlinear dynamics in Kerr medium. We can equivalently rewrite the nonlinear transformation operators originated from the Hamiltonian $H = \hbar\chi N(N-1)$ in the normal ordered form which can be easily dealt with. We also give some applications of this approach.

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1. Introduction

Kerr medium [1] plays a key role in the applications of nonlinear quantum optics, for example, nonlinear Mach–Zehnder interferometer [2], quantum nondemolition measurement [3], the generation of macroscopic quantum superposition states [4, 5], holonomic quantum computation within quantum optics [6] and so on. Recently Kerr medium also finds its application in the generation of multi-dimensional entangled coherent states [7, 8].

As is well known, the interaction between one-mode electromagnetic field and Kerr medium can be described by the Hamiltonian [1]

$$H = \hbar\chi N(N-1), \tag{1}$$

where χ is a factor proportional to the third-order nonlinear susceptibility of the medium and N is the number operator describing the mode propagating through the Kerr medium. Due to the nonlinear relations between the Hamiltonian H and number operator N, it is usually hard to sum the relevant series which express the nonlinear transformations in various calculations of time evolutions and quantum statistics [9, 10]. In [4, 7] the nonlinear evolution $\exp(-\mathrm{i}tH/\hbar)$ can be expressed by a sum of linear transformations when the continuous parameter t assumes a series of given values such that the time-evolution operator is periodic with respect to the number operator N, from which they obtain the multi-dimensional coherent states. Generally, the problem to analytically calculate the nonlinear dynamics of Kerr medium in the case of continuously or imaginary parameter t was not treated of in [4, 7] yet.

Based on the nonlinear map defined in this paper, we present a new calculation approach in which the nonlinear transformation operator including N^2 in the exponent can be transformed identically into a linear transformation operator under the normal ordered form including N in the exponent. As applications, we can calculate the time-evolution case with continuously varied imaginary parameter t in the exponent which enables an analytical calculation of the dynamical evolution of arbitrary initial states in a simple way, and we also derive the quantum partition function of Kerr medium where the parameter t in the exponent is imaginary.

This paper is organized as follows: section 2, which is the main part of the paper, includes a basic deduction of equation (5a) and some discussions. In section 3, the analytical expressions of the final states evolved from arbitrary initial states are given. In section 4, in order to show that our results may contain those given in [4, 7] for some given time values, we deduce their results with a more concise approach with correct coefficients. In section 5, we consider the quantum partition function of Kerr medium. Section 6 is devoted to concluding remarks. In the appendix we generalize our approach to a general nonlinear medium with an $f(N)$ interaction.

2. The basic formula of the operator map of the Kerr medium and some discussions

First we introduce a nonlinear map defined as follows:

$$\langle x\rangle^n \equiv x^{n^2}, \qquad x \in C. \tag{2}$$

This notation of the angle bracket prescribes a nonlinear map of the power of variable x. We define a function $f(\langle x\rangle)$ of $\langle x\rangle$ according to its power series assuming that $f(x)$ is an enough regular function. For example

$$\mathrm{e}^{\langle x\rangle} \equiv \sum_{n=0}^{\infty} \frac{1}{n!}\langle x\rangle^n \equiv \sum_{n=0}^{\infty} \frac{1}{n!} x^{n^2}, \tag{3}$$

and for another example

$$(\langle \mathrm{e}^{\lambda}\rangle - 1)^n \equiv \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \mathrm{e}^{k^2\lambda}, \tag{4}$$

where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$. With above notations, we can prove the following *basic* formula:

$$\exp(\lambda N^2) =: \exp\{(\langle \mathrm{e}^{\lambda}\rangle - 1)N\}:, \qquad \lambda \in C, \tag{5a}$$

which holds true for arbitrary complex constant λ.

Proof. To prove equation (5a), we only need to prove the following expression:

$$\mathrm{e}^{\lambda N^2} = \sum_{l=0}^{\infty} c_l (a^+)^l a^l, \tag{5b}$$

□

with the coefficients c_l given by

$$c_l \equiv \frac{1}{l!}\{\langle \mathrm{e}^{\lambda}\rangle - 1\}^l = \frac{1}{l!} \sum_{k=0}^{l} (-1)^{l-k} \binom{l}{k} \mathrm{e}^{\lambda k^2}. \tag{6}$$

Now we prove equation (6) by induction. In fact, if we let the two sides of the expanding expressions (5b) act on a number state $|k\rangle$, we can obtain $\mathrm{e}^{\lambda k^2}$ and $\sum_{l=0}^{k} l!\binom{k}{l} c_l$ for the left

side and right side, respectively. Now we first prove that equation (6) is correct for the cases of $k = 0, 1, 2$:
for $k = 0$, $c_0 = 1$;
for $k = 1$, $\mathrm{e}^{\lambda} \equiv x = c_0 + c_1$;
then we get $c_1 = \mathrm{e}^{\lambda} - 1 \equiv \{\langle x\rangle - 1\}$; and the equality is also correct for $k = 2$ with $c_2 = \frac{1}{2!}\{\langle x\rangle - 1\}^2$. Then, we assume the formula holds for $c_0, \ldots, c_k$ and establish it for c_{k+1}. Let the two sides of the expanding expressions equation (5b) act on the number state $|k+1\rangle$, we get

$$\mathrm{e}^{(k+1)^2\lambda} = \sum_{l=0}^{k+1} l!\binom{k+1}{l} c_l. \tag{7}$$

We may get the coefficient c_{k+1} from the above equation

$$\begin{aligned}
c_{k+1} &= \frac{1}{(k+1)!}\left\{\mathrm{e}^{(k+1)^2\lambda} - \sum_{l=0}^{k} l!\binom{k+1}{l} c_l\right\} \\
&= \frac{1}{(k+1)!}\left\{\mathrm{e}^{(k+1)^2\lambda} - \sum_{l=0}^{k} l!\binom{k+1}{l} [\langle \mathrm{e}^{\lambda}\rangle - 1]^l\right\} \\
&= \frac{1}{(k+1)!}\left\{\mathrm{e}^{(k+1)^2\lambda} - \binom{k+1}{k}\binom{k}{k}\mathrm{e}^{k^2\lambda} + \left[\binom{k+1}{k}\binom{k}{k-1}\right.\right. \\
&\quad \left. - \binom{k+1}{k-1}\binom{k-1}{k-1}\right]\mathrm{e}^{(k-1)^2\lambda} + \left[-\binom{k+1}{k}\binom{k}{k-2} + \binom{k+1}{k-1}\binom{k-1}{k-2}\right. \\
&\quad \left.\left. - \binom{k+1}{k-2}\binom{k-2}{k-2}\right]\mathrm{e}^{(k-2)^2\lambda} + \cdots + \sum_{l=0}^{k}(-1)^{l+1}\binom{k+1}{l}\right\}.
\end{aligned} \tag{8}$$

It is easy to know that the last term of the above equation is equal to $(-1)^{k+1}$. At last we get

$$c_{k+1} = \frac{(-1)^{k+1}}{(k+1)!}\sum_{l=0}^{k+1}(-1)^l\binom{k+1}{l}\mathrm{e}^{\lambda l^2}. \tag{9}$$

The proof is complete.

From equation (5a) we see that a nonlinear transformation can appear linear in its normal ordered form. This provides a counter example of the statement that quadratic Hamiltonian is the highest order solvable in closed form with normal ordering method [2]. Apparently, equation (5a) can be trivially generalized to multimode without coupling

$$\mathrm{e}^{\lambda\sum_i N_i^2} =: \mathrm{e}^{(\langle \mathrm{e}^{\lambda}\rangle - 1)\sum_i N_i} : . \tag{10}$$

Equation (5a) also can be generalized to the case of arbitrary function (see details in the appendix).

We should also pay attention to the following points while we deal with above nonlinear map. At first, we cannot simply sum up their powers when two angle brackets multiply, i.e., $\langle x\rangle^n \cdot \langle x\rangle^m \neq \langle x\rangle^{n+m}$. As a result we should note

(1) $$\begin{aligned} : \mathrm{e}^{[(\langle \mathrm{e}^{\lambda}\rangle - 1)N]} : a : \mathrm{e}^{[(\langle \mathrm{e}^{-\lambda}\rangle - 1)N]} &:= \mathrm{e}^{N\ln\langle \mathrm{e}^{\lambda}\rangle} a\, \mathrm{e}^{N\ln\langle \mathrm{e}^{-\lambda}\rangle} \\ &\neq \mathrm{e}^{-\ln\langle \mathrm{e}^{\lambda}\rangle} a = \mathrm{e}^{-\lambda} a. \end{aligned} \tag{11}$$

We should adopt the following calculation:

$$\begin{aligned}\mathrm{e}^{(\ln\langle \mathrm{e}^{\lambda}\rangle)N} a\, \mathrm{e}^{-(\ln\langle \mathrm{e}^{\lambda}\rangle)N} &= \mathrm{e}^{(\ln\langle \mathrm{e}^{\lambda}\rangle)N} \mathrm{e}^{-(\ln\langle \mathrm{e}^{\lambda}\rangle)(N+1)} a \\ &= \mathrm{e}^{-\lambda(2N+1)} a,\end{aligned} \tag{12}$$

from the following identity

$$\mathrm{e}^{-f(N)} a\, \mathrm{e}^{f(N)} = \mathrm{e}^{f(N+1)-f(N)} a. \tag{13}$$

(2) The expression of the inverse operator should be

$$\{: \exp[(\langle x\rangle - 1)N] :\}^{-1} =: \exp\left[\left(\left\langle \frac{1}{x}\right\rangle - 1\right) N\right] : (x \neq 0). \tag{14}$$

(3) Three functions $\mathrm{e}^{\langle x\rangle}$, $\langle \mathrm{e}^{x}\rangle$ and $\langle \mathrm{e}\rangle^{x}$ are different from each other. Secondly, after getting rid off the normal ordered form, equation (5*a*) is equal to

$$: \exp[(\langle \mathrm{e}^{\lambda}\rangle - 1)N] := \exp[(\ln\langle \mathrm{e}^{\lambda}\rangle)N]. \tag{15}$$

Actually, with above notations we can easily show that equation (15) holds according to equation (5*a*) because the left side of equation (15) is

$$\mathrm{e}^{\lambda N^2} \equiv \langle \mathrm{e}^{\lambda}\rangle^{N} = [\exp(\ln\langle \mathrm{e}^{\lambda}\rangle)]^{N} = \exp[(\ln\langle \mathrm{e}^{\lambda}\rangle)N]. \tag{16}$$

3. The analytical solution to the dynamic evolution in Kerr medium

Although the analytical calculations of the nonlinear transformation related to Kerr medium in simple cases can be straightforward completed, it is usually hard to deal with correlated calculations of Kerr medium in the case of continuous parameter or imaginary time t due to the nonlinear property [10]. In the following we calculate analytically the dynamic evolution from arbitrary initial states using results obtained previously.

For example, suppose that the initial state is $|\varphi(0)\rangle = \varphi(a^{\dagger})|0\rangle$, then the resulting state at time t is

$$|\varphi(t)\rangle =: \mathrm{e}^{(\langle \mathrm{e}^{-\mathrm{i}\chi t}\rangle - 1)N} : \varphi(a^{\dagger}\mathrm{e}^{\mathrm{i}\chi t})|0\rangle. \tag{17a}$$

On the other hand, inserting the completeness of coherent states we obtain further

$$\begin{aligned}|\varphi(t)\rangle &= \int \frac{\mathrm{d}^2 z}{\pi} : \mathrm{e}^{(\langle \mathrm{e}^{-\mathrm{i}\chi t}\rangle - 1)N} : |z\rangle\langle z|\varphi(a^{\dagger}\,\mathrm{e}^{\mathrm{i}\chi t})|0\rangle \\ &= \int \frac{\mathrm{d}^2 z}{\pi}\, \mathrm{e}^{-|z|^2 + \langle \mathrm{e}^{-\mathrm{i}\chi t}\rangle a^{\dagger} z} \varphi(z^*\,\mathrm{e}^{\mathrm{i}\chi t})|0\rangle.\end{aligned} \tag{17b}$$

Next we calculate an example when $|\varphi(0)\rangle = |3\rangle$,

$$\begin{aligned}\int \frac{\mathrm{d}^2 z}{\pi}\, \mathrm{e}^{-|z|^2 + \langle \mathrm{e}^{-\mathrm{i}\chi t}\rangle a^{\dagger} z} \frac{(z^*)^3}{\sqrt{6}}\, \mathrm{e}^{3\mathrm{i}\chi t}|0\rangle &= \frac{\mathrm{e}^{3\mathrm{i}\chi t}}{\sqrt{6}} \partial_{\beta}^{3} \int \frac{\mathrm{d}^2 z}{\pi}\, \mathrm{e}^{-|z|^2 + \langle \mathrm{e}^{-\mathrm{i}\chi t}\rangle a^{\dagger} z + z^*\beta}|0\rangle|_{\beta=0} \\ &= \mathrm{e}^{-6\mathrm{i}\chi t}|3\rangle.\end{aligned} \tag{18}$$

Obviously, the result is the same to that of equation (17*a*). Therefore, by means of this approach and the completeness of coherent states we have resolved generally the nonlinear dynamical calculations of Kerr medium in a closed form.

4. Some applications of the time evolution operator of Kerr medium

For some given time series $t_m = \frac{\pi}{m\chi}$ with m being positive integer numbers, the time evolution operator of Kerr medium is simplified as [4]

$$U(t) = \mathrm{e}^{-\mathrm{i}\chi t N(N-1)} = \mathrm{e}^{-\mathrm{i}\frac{\pi}{m}N(N-1)}. \tag{19}$$

Therefore, $\exp[-\mathrm{i}(\pi/m)N^2]$ is a periodic function with period m when m = even; $\exp[-\mathrm{i}(\pi/m)N(N-1)]$ is also a periodic function with period m when m = odd. For these two periodic functions we can expand them as discrete Fourier series respectively

$$\exp\left(\frac{-\mathrm{i}\pi}{m}N^2\right) = \sum_{q=0}^{m-1} f_q^{(e)} \exp\left(\frac{-\mathrm{i}2\pi q}{m}N\right), \tag{20a}$$

with $f_q^{(e)} = \frac{1}{\sqrt{m}}\exp(\mathrm{i}\pi q^2/m - \mathrm{i}\pi/4)$ for m = even;

$$\exp\left[\frac{-\mathrm{i}\pi}{m}N(N-1)\right] = \sum_{q=0}^{m-1} f_q^{(o)} \exp\left(\frac{-\mathrm{i}2\pi q}{m}N\right), \tag{20b}$$

with $f_q^{(o)} = \frac{1}{\sqrt{m}}\exp[\mathrm{i}\pi q(q+1)/m - \mathrm{i}\pi(m-1)/4m]$ for m = odd.

At first we prove equation (20a). From equation (5a) we know

$$\mathrm{e}^{\frac{-\mathrm{i}\pi}{m}N^2} = \sum_{n=0}^{\infty}\frac{(a^+)^n a^n}{n!}\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\mathrm{e}^{-\mathrm{i}\pi k^2/m}. \tag{21}$$

In order to expand each phase factor in brace as the discrete Fourier series we use the following formula [11]:

$$\sum_{q=0}^{k-1}\mathrm{e}^{\frac{\mathrm{i}2\pi q^2}{k}} = \frac{\sqrt{k}}{2}(1+\mathrm{i})\left(1+\mathrm{e}^{\frac{\mathrm{i}k\pi}{2}}\right). \tag{22}$$

Moreover, let $k = 2m$, m = even, we get

$$\sum_{q=0}^{m-1}\mathrm{e}^{\frac{\mathrm{i}\pi q^2}{m}} = \sqrt{m}\,\mathrm{e}^{\frac{\mathrm{i}\pi}{4}}. \tag{23}$$

Then, let $n = lm + q'$, $0 \leqslant q' \leqslant m-1$, we have

$$\begin{aligned}\sum_{q=0}^{m-1}\mathrm{e}^{\frac{\mathrm{i}\pi(q-n)^2}{m}} &= \sum_{q=0}^{m-1}\mathrm{e}^{\frac{\mathrm{i}\pi(q-q')^2}{m}}\\ &= \sum_{q=0}^{m-1}\mathrm{e}^{\frac{\mathrm{i}\pi q^2}{m}} = \sqrt{m}\,\mathrm{e}^{\frac{\mathrm{i}\pi}{4}}.\end{aligned} \tag{24}$$

The second equality sign holds due to the fact that the only difference of these two sums is the sequence of terms in sums. Using equation (24) we can rewrite equation (21) as

$$\begin{aligned}\mathrm{e}^{\frac{-\mathrm{i}\pi}{m}N^2} &= \sum_{n=0}^{\infty}\frac{1}{n!}\frac{1}{\sqrt{m}}\mathrm{e}^{\frac{-\mathrm{i}\pi}{4}}(a^\dagger)^n a^n\cdot\sum_{l=0}^{n}(-1)^{n-l}\binom{n}{l}\left(\sum_{q=0}^{m-1}\mathrm{e}^{\frac{\mathrm{i}\pi(q-l)^2}{m}}\right)\mathrm{e}^{\frac{-\mathrm{i}\pi l^2}{m}}\\ &= \frac{1}{\sqrt{m}}\sum_{q=0}^{m-1}\mathrm{e}^{\frac{\mathrm{i}\pi q^2}{m}-\frac{\mathrm{i}\pi}{4}} :\mathrm{e}^{(\mathrm{e}^{\frac{-\mathrm{i}2\pi q}{m}}-1)N}: .\end{aligned} \tag{25}$$

The last expression of the above equation is just equation (20a).

In the case of odd m, which is not shown in details in [7], we use the identity

$$\sum_{q=0}^{m-1} \mathrm{e}^{\frac{\mathrm{i}\pi(q+\frac{1}{2})^2}{m}} = \sqrt{m}\,\mathrm{e}^{\frac{\mathrm{i}\pi}{4}}, \tag{26}$$

then it is easy to prove the following formula:

$$\mathrm{e}^{\frac{-\mathrm{i}\pi}{m}N(N-1)} = \frac{1}{\sqrt{m}} \sum_{q=0}^{m-1} \mathrm{e}^{\frac{\mathrm{i}\pi q(q+1)}{m} - \frac{\mathrm{i}\pi(m-1)}{4m}}\, \mathrm{e}^{-\frac{\mathrm{i}2\pi q}{m}N}. \tag{27}$$

This is just equation (20b). The proof is complete.

It should be pointed out that the coefficients $f_q^{(o)} = \frac{1}{\sqrt{m}} \exp[\mathrm{i}\pi q(q+1)/m - \mathrm{i}\pi k(k+1)/4]$ with $m = 2k+1$ listed in [7] in the case of odd m are wrong.

We note that in fact we can obtain the expanding expression equation (20a) and equation (20b) directly, taking into account the following two identities:

$$\mathrm{e}^{\frac{-\mathrm{i}\pi n^2}{m}} = \frac{\mathrm{e}^{-\mathrm{i}\pi/4}}{\sqrt{m}} \sum_{q=0}^{m-1} \mathrm{e}^{\frac{\mathrm{i}\pi q^2}{m} - \frac{\mathrm{i}2\pi q n}{m}}, \tag{28a}$$

for $m =$ even;

$$\mathrm{e}^{\frac{-\mathrm{i}\pi n(n-1)}{m}} = \frac{\mathrm{e}^{\frac{-\mathrm{i}\pi}{4m}(m-1)}}{\sqrt{m}} \sum_{q=0}^{m-1} \mathrm{e}^{\frac{\mathrm{i}\pi q(q+1)}{m} - \frac{\mathrm{i}2\pi q n}{m}}, \tag{28b}$$

for $m =$ odd. If we let equation (28a) and equation (28b) multiply with $|n\rangle$ and replace n by operator N, and act on $\langle n|$, then we will get the expanding results equation (20a) and equation (20b) after taking into account the completeness $\sum_{n=0}^{\infty} |n\rangle\langle n| = I$. From this it seems that we can prove equation (5a) by equation (28a) and equation (28b). Actually, it is not the fact because equation (5a) includes the case of real value exponent. The above deduction from equation (28a) and equation (28b) verifies equation (5a) in the case of imaginary exponent only.

Now let us consider a usual experimental scheme in which a laser is mixed with the vacuum state on beam splitters after passing through a Kerr medium. Dynamics is described by the following unitary evolution operator [7]:

$$U_{A,B}(\tau) = \mathrm{e}^{\frac{\pi}{4}(a^\dagger b - b^\dagger a)}\, \mathrm{e}^{-\mathrm{i}\tau (a^\dagger)^2 a^2}, \tag{29}$$

where τ is the time spent inside the kerr medium. According to the above formula, with a concise approach we can get the multi-dimensional entangled coherent state [7] generated by this unitary operator $U_{A,B}(\tau)$,

$$|\Phi_m\rangle = \sum_{q=0}^{m-1} f_q^{(o)} |\alpha\,\mathrm{e}^{-2\pi\mathrm{i}q/m}\rangle |\alpha\,\mathrm{e}^{-2\pi\mathrm{i}q/m}\rangle, \tag{30}$$

for m odd with $\alpha = \beta/\sqrt{2}$, and

$$|\Phi_m\rangle = \sum_{q=0}^{m-1} f_q^{(e)} |\alpha\,\mathrm{e}^{-2\pi\mathrm{i}q/m}\rangle |\alpha\,\mathrm{e}^{-2\pi\mathrm{i}q/m}\rangle, \tag{31}$$

for m even with $\alpha = \beta \exp(\mathrm{i}\pi/m)/\sqrt{2}$.

5. The quantum partition function of Kerr medium

Now we investigate the quantum statistics of Kerr medium, i.e, the case with λ in equation (5*a*) being real number. The eigenvalues and eigenvectors of this Hamiltonian are well known in Fock space. But it is hard to sum up the relevant series. Even using the Possion's summation formula [12], we cannot get the analytical expression of the partition function and have to resort to numerical calculations. However, with the present approach the quantum partition function can be expressed in terms of a compact and analytical form

$$Z[\beta] = \sum_{n=0}^{\infty} \mathrm{e}^{-\alpha\beta n(n-1)} = [1 - \langle \mathrm{e}^{-\alpha\beta} \rangle\, \mathrm{e}^{\alpha\beta}]^{-1}, \tag{32}$$

where $\alpha = \hbar\chi$ and $\beta = 1/kT$. Obviously, equation (32) may be obtained directly with the approach of the angle bracket. But here we present a different derivation of equation (32)

$$\begin{aligned} Z[\beta] &= \int \frac{\mathrm{d}^2 z}{\pi} \langle z|\, \mathrm{e}^{\alpha\beta N} :\mathrm{e}^{(\langle \mathrm{e}^{-\alpha\beta}\rangle - 1)N} : |z\rangle \\ &= \int \frac{\mathrm{d}^2 z}{\pi} \mathrm{e}^{(\langle \mathrm{e}^{-\alpha\beta}\rangle - 1)\mathrm{e}^{\alpha\beta}|z|^2} \langle z| : \mathrm{e}^{(\mathrm{e}^{\alpha\beta}-1)N} : |z\rangle \\ &= [1 - \langle \mathrm{e}^{-\alpha\beta}\rangle\, \mathrm{e}^{\alpha\beta}]^{-1}. \end{aligned} \tag{33}$$

6. Conclusions

The central result of this paper, i.e., equation (5*a*), shows how we can transform the nonlinear evolution operator of Kerr medium equivalently into a linear transformation expression in the normal ordered form. We present an alternative and apparently better way in performing explicit analytic calculations involving the Kerr medium. This approach has two advantages: its linearity and its capacity of dealing with both real and imaginary, discrete and continuous parameters. As examples, we have obtained in a compact form the final state of Kerr medium from arbitrary initial states after a given time, the expressions of multi-dimensional entangled coherent states for both odd and even cases with correct coefficients and a compact form of quantum partition function of Kerr medium.

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Appendix

Equation (5*a*) can be generalized to arbitrary function

$$\exp[\lambda f(N)] =: \exp\{(\langle \mathrm{e}^{\lambda}\rangle_f - 1)N\} :, \tag{A.1}$$

where $\langle \mathrm{e}^{\lambda}\rangle_f^n = \mathrm{e}^{\lambda f(n)}$. We give another simple proof which is different from the proof of equation (5*a*)

Proof. In order to prove equation (A.1) we just need to prove the following equality:

$$\mathrm{e}^{\lambda f(N)} = \sum_{l=0}^{\infty} c_l (a^+)^l a^l, \tag{A.2}$$

with coefficients c_l,

$$c_l \equiv \frac{1}{l!}\{\langle e^{\lambda}\rangle_f - 1\}^l. \tag{A.3}$$

From the equality

$$\begin{aligned}\sum_{l=s}^{k} \frac{(-1)^{k-l}}{(l-s)!(k-l)!} &= \begin{cases} 1 & \text{if } k=s \\ 0 & \text{if } k>s \end{cases} \\ &= \delta_{kl},\end{aligned} \tag{A.4}$$

we get

$$\begin{aligned}F(k) &= \sum_{s=0}^{k} F(s)\delta_{ks} = \sum_{s=0}^{k} F(s) \sum_{l=s}^{k} \frac{(-1)^{l-s}}{(l-s)!(k-l)!} \\ &= \sum_{l=0}^{k} \frac{1}{(k-l)!} \sum_{s=0}^{l} F(s) \frac{(-1)^{l-s}}{(l-s)!}.\end{aligned} \tag{A.5}$$

$F(k)$ is an arbitrary function. Now, let $F(k) = \frac{e^{\lambda f(N)}}{k!}$ we have

$$\begin{aligned}e^{\lambda f(N)} &= \sum_{l=0}^{k} l! \binom{k}{l} \frac{(-1)^{l-s}}{s!(l-s)!} e^{\lambda f(s)} \\ &= \sum_{l=0}^{k} l! \binom{k}{l} \frac{1}{l!} (\langle e^{\lambda}\rangle_f - 1)^l.\end{aligned} \tag{A.6}$$

□

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